Math 821, Spring 2013, Lecture 5

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1 A Taste of Bivariate Generating Functions

Let $\mathcal{P}$ be a combinatorial class of permutations.

$$\mathcal{P} = Set(DCyc_{labelled}(\mathcal{Z})) - Seq_{unlabelled}(\mathcal{Z})$$

So

$$P(x) = \exp\left(\log\left(\frac{1}{1-x}\right)\right) = \frac{1}{1-x}$$

Let's restrict: Let A,B be subsets of $\mathbb{Z}_{\geq 0}$ Let $\mathcal{P}^{(A,b)}$ be the class of permutations with cycle lengths in A and the number of cycles in B

$$\mathcal{P}^{(A,b)} = Set_B\left(DCyc_A\left(\mathcal{Z}\right)\right)$$

So

$$P(x) = \sum_{b\in B}\frac{1}{b!}\left(\sum_{a\in A}\frac{x^a}{a}\right)^b$$

An important special instance of this is in permutations consisting of exactly r cycles, $\mathcal{P}^{(\mathbb{Z}_{\geq 0},r)}$. So

$$P^{(\mathbb{Z}_{\geq 0},r)}(x) = \frac{1}{r!}\left(\log\frac{1}{1-x}\right)^r$$

Definition. The number of permutations of n with exactly r cycles is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{r!}[x^n]\left(\log\frac{1}{1-x}\right)^r$$

these are called Stirling numbers of the first kind.

Now suppose we want to know how many cycles are in a random permutation. By random, we mean that among all permutations of size n, each is equally likely. Thus the probability that a random permutation of size n has k cycles is $p_{n,k} = \frac{1}{n!}\begin{bmatrix} n \\ r \end{bmatrix}$.

For n = 100:

k	1	2	3	4	5	6	7	8	9	10
$p_{n,k}$	0.01	0.05	0.12	0.19	0.21	0.17	0.11	0.06	0.03	0.01

But how does the expected number of cycles grow as n grows? We can answer this with bivariate generating functions.

Define

$$P(x,y) = \sum_{\sigma \in \mathcal{P}} \frac{x^{|\sigma|}}{|\sigma|!} y^{(\#cycles of \sigma)}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} y^k$$

$$= \sum_{r=0}^{\infty} P^{(\mathbb{Z}_{\geq 0}, r)}(x) y^r$$

(Note: this generating function is an exponential generation function in x, but an ordinary generating function in y)

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \left(\log \frac{1}{1-x} \right)^r y^r$$

$$= \exp\left(y \log \frac{1}{1-x} \right)$$

$$= \exp\left(\log \left(\frac{1}{1-x} \right)^y \right)$$

$$= (1-x)^{-y}$$

which is nice enough that we can usefully manipulate it.

Now, the expected number of cycles in a permutation of n is (recall $\sum xP(X = x)$)

$$E_n = \sum_{r=0}^{\infty} r p_{n,r}$$

Now

$$[x^n]P(x, y) = [x^n](1 - x)^{-y}$$
$$= (-1)^n \left(\begin{array}{c} -y \\ n \end{array} \right)$$
$$= \frac{(-1)^n(-y)(-y-1)\cdots(-y-n+1)}{n!}$$
$$= \frac{y(y+1)\cdots(y+n-1)}{n!}$$

As well:

$$[x^n]P(x, y) = \sum_{r=0}^{\infty} \left[\begin{array}{c} n \\ r \end{array} \right] \frac{y^r}{n!}$$
$$= \sum_{r=0}^{\infty} p_{n,r} y^r$$

So take a derivative, and set y = 1.

$$E_n = \sum_{r=0}^{\infty} r p_{n,r} = \sum_{i=0}^{n-1} \left. \frac{(y)\cdots(y+i-1)(1)(y+i+1)\cdots(y+n-1)}{n!} \right|_{y=1}$$
$$= \sum_{i=0}^{n-1} \frac{1(2)\cdots(i)(1)(i+2)\cdots(n)}{n!}$$
$$= \sum_{i=0}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n} \frac{1}{i}$$

In general, $E_n \sim \log n$ as $n \to \infty$.

References. Flajolet and Sedgewick, *Analytic Combinatorics*, Cambridge (2009). II.4.

2 Introduction to Asymptotic Analysis

Definition. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be sequences. Then we write $a_n \sim b_n$ as $n \to \infty$ to mean

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

We say a_n and b_n are asymptotically equal.

Often, you'll find $a_n \sim c\rho^n Poly(n)$.

Definition. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be sequences. Then we write $a_n \bowtie b_n$ as $n \to \infty$ to mean

$$\limsup_{n\to\infty} a_n^{\frac{1}{n}} = \limsup_{n\to\infty} b_n^{\frac{1}{n}}$$

and we say (a_n) and (b_n) have the same exponential growth.

Example. Say $a_n = \alpha^n(1 + f(n))$ where $f(n) \geq 0$ and $f(n)$ grows strictly slower than any exponential i.e.

$$\lim \frac{f(n)}{\beta^n} \to 0 \quad \forall \beta{>}0$$

Then $a_n \bowtie \alpha^n$ since

$$a_n^{\frac{1}{n}} = \alpha(1 + f(n))^{\frac{1}{n}} \Rightarrow (\alpha^n)^{\frac{1}{n}} = \alpha$$

Suppose $(1 + f(n))^{\frac{1}{n}} \geq c > 1$ infinitely often. Then $1 + f(n) \geq c^n$ So

$$\frac{1 + f(n)}{c^n} > 1$$

Which is a contradiction. So

$$\lim_{n\to\infty}(1 + f(n))^{\frac{1}{n}} = 1 \Rightarrow \limsup_{n\to\infty} a_n^{\frac{1}{n}} = \alpha = \limsup_{n\to\infty}(\alpha_n)^{\frac{1}{n}} \Rightarrow a_n \bowtie \alpha^n$$

Definition. Let Ω be a connected open subset of $\mathbb{C}$. A function $f : \Omega \to f$ is *analytic* at a point $z_0 \in \Omega$ if for some open disc centered at z_0 and within Ω, $f(a)$ is represented with a convergent power series. We say f is analytic in Ω if it is analytic at every $z_0 \in \Omega$.

Definition. Let Ω be a connected open subset of $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is *holomorphic* at $z_0 \in \Omega$ if

$$\lim_{\delta \to 0} \frac{f(z_0 + \delta) - f(z_0)}{\delta}$$

exists independently of the path δ takes to 0. This will be $f'(z_0)$.
f is holomorphic on Ω if it is holomorphic at every point of Ω.

Theorem. Let Ω be as described, and $f : \Omega \to \mathbb{C}$. Then f is holomorphic on Ω if and only if f is analytic on Ω. And if so, then f is smooth on Ω (i.e. all its derivatives exist).

Definition. Let Ω be a connected open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be analytic on Ω and let z_0 be a point on the boundary of Ω. If there exists an analytic function g on a connected open set Ω' such that $z_0 \in \Omega'$ and $f = g$ on the intersection $\Omega \cap \Omega'$, then we say g is an *analytic continuation* of f.

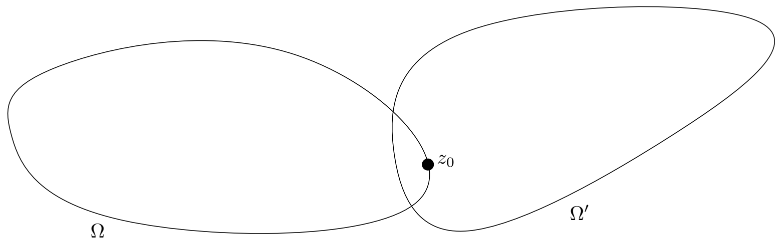

Fig. 1

Theorem. The analytic continuation of an analytic function is unique.

Defintion. Let Ω be a connected open subset of $\mathbb{C}$, and $f : \Omega \to \mathbb{C}$ be analytic. A point z_0 on the boundary of Ω is a *singularity* of f if f is not analytically continuable at z_0.

The only singularities we are going to care about are the ones on the circle of convergence of power series with nonnegative coefficients.

Theorem(**Pringsheim's**). Let $f(z)$ be analytic at zero with a series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

Suppose each $f_i \geq 0$ and the series has radius of convergence $0 < R < \infty$, Then $z = R$ is a singularity of $f(z)$ and no singularity with smaller norm exists.

Intuition. A real and positive x is no smaller than an x not on the real axis, but with the same norm.

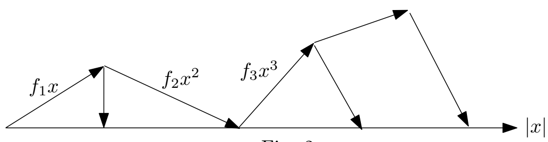

Fig. 2

Proof. The series has radius R and matches the function, so there are no singularities of norm $< R$. Suppose on the contrary that f is analytic at $R = z$. Then f is analytic in a disc of radius r centered at $z = R$. Choose $0 < h < \frac{1}{3}r$ and consider the expansion of f around $R - h$

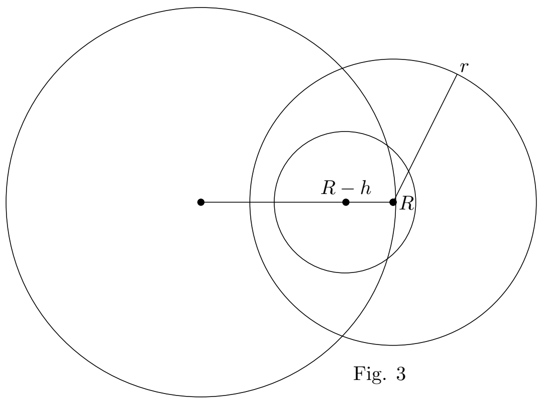

Fig. 3

$$f(z) = \sum_{m \geq 0} g_m (z - z_0)^m$$

In the intersection of circles

$$\sum_{n=0}^{\infty} f_n z^n = \sum_{m=0}^{\infty} g_m (z - z_0)^m$$

pull out g_m by taking m derivatives at z_0, and get

$$g_m = \sum_{n=0}^{\infty} \begin{pmatrix} n \\ m \end{pmatrix} f_n z_0^{n-m}$$

and so the g_m are also nonnegative.
Now $f(z) = \sum_{m \geq 0} g_m (z - z_0)^m$ converges at $R + h$ by choice of h. So

$$f(R + h) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \begin{pmatrix} n \\ m \end{pmatrix} f_n z_0^{n-m} \right) (2h)^m$$

this is a convergent series with all nonnegative terms.

We can rearrange to get

$$f(R+h) = \sum_{n=0}^{\infty} f_n \left(\sum_{m=0}^{\infty} \left(\begin{array}{c} n \\ m \end{array} \right) z_0^{n-m}(2h)^m \right)$$

$$= \sum_{n=0}^{\infty} f_n(z_0+2h)^n$$

where $z_0 + 2h = R + h$, contradicting that the series has radius of convergence of R. ■

Definition. Singularities on the boundary of the disc of convergence of a power series are called *dominant singularities*.

Example. Consider the binary rooted tree with distinct left and right childre from the first day. $\mathcal{T} = \mathcal{E} + \mathcal{Z} \times \mathcal{Z}^2$.
So $T(x) = 1 + xT(x)^2$, so $T(x) = \frac{1-\sqrt{1-4x}}{2x}$.
What is a dominant singularity? At $x = \frac{1}{4}$, when the square root is about to be negative. By Pringsheim, we only need to look at real possible x values. So $x = \frac{1}{4}$ is a real dominant singularity and $\frac{1}{4}$ is the radius of convergence.

The First Principle of Combinatorial Asymptotics
The radius of convergence, or position of dominant singularity on $\mathbb{R}_{>0}$, determines the exponential growth.

This is encapsulated in the following:

Proposition. Let $f(z)$ be analytic at 0 with the series expansion $f(z) = \sum_{n=0}^{\infty} f_n z^n$, with the $f_i \geq 0$ and radius of convergence $0 < R < \infty$. Then $f_n \bowtie \left(\frac{1}{r}\right)^n$.

Proof. By definition of radius of convergence, $\forall \epsilon > 0$, $\sum_{n=0}^{\infty} f_n(R-\epsilon)^n$ converges. So in particular, $f_n(R-\epsilon)^n \to 0$ as $n \to \infty$, and in particular $f_n(R-\epsilon)^n < 1$ for n sufficiently large.

$$f_n < \frac{1}{(R-\epsilon)^n}$$

for n sufficiently large. Thus,

$$f_n^{\frac{1}{n}} < \frac{1}{R-\epsilon}$$

In the other direction, $f_n(R+\epsilon)^n$ is unbounded since $\sum f_n(R+\epsilon)^n$ is strictly outside the radius of convergence. So

$$
\begin{aligned}
f_n(R+\epsilon)^n &> 1 \\
f_n &> \frac{1}{(R+\epsilon)^n} \\
f_n^{\frac{1}{n}} &> \frac{1}{R+\epsilon}
\end{aligned}
$$

infinitely often. Therefore,

$$\limsup f_n^{\frac{1}{n}} = \frac{1}{R+\epsilon}$$

so

$$f_n \bowtie \left(\frac{1}{R}\right)^n$$

■

Example. Take $\mathcal{T}$ from the previous example. $T(x) = \frac{1-\sqrt{1-4x}}{2x}$.
The dominant singularity on $\mathbb{R}_{>0}$ is $x = \frac{1}{4}$, so $t_n \bowtie 4^n$.

References. Flajolet and Sedgewick, *Analytic Combinatorics*, Cambridge (2009). IV.2, IV.3.